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1999 J. Phys. A: Math. Gen. 32 955

(http://iopscience.iop.org/0305-4470/32/6/009)

J. Phys. A: Math. Gen. **32** (1999) 955–976. Printed in the UK PII: S0305-4470(99)95278-2

Change of variables as Borel resummation of semiclassical series

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Received 18 June 1998, in final form 3 November 1998

Abstract. The 1D Schrödinger equation with meromorphic potentials are considered. It is shown that any fundamental solution constructed for such potentials has a property to depend in some particular way on an arbitrary meromorphic function $f(x)$. A family of fundamental solutions created in this way can be reproduced by multiplying some fixed member of the family by appropriate $\hbar$-dependent constants. This freedom allows for a proper construction of fundamental solutions at simple and double poles of considered potentials. It expresses the fact that any change of variable $x \to y = y(x)$ keeping the form of 1D Schrödinger equation keeps invariant the form of the fundamental solutions as well. In fact there is one-to-one correspondence between the functions $f(x)$ and the Schwarzian of $y(x)$. Since any fundamental solution is Borel summable it means also that the effect of the change of variable in fundamental solutions can be recoverd by multiplying the 'initial' fundamental solutions by suitably chosen $\hbar$-dependent constants and Borel resumming the corresponding semiclassical expansions. This explains why a change of variable can improve JWKB formulae. However, it is shown also that a change of variable itself cannot provide us with the exact JWKB formulae.

1. Introduction

A change of variable in the 1D Schrödinger equation is one of the basic techniques used to solve 1D problems (see [12], for example). In the context of semiclassical (JWKB) approximation the procedure is in fact a main ingredient of Fröman and Fröman's approach to the 1D Schrödinger equation [3, 4] with the aim of getting improved JWKB quantization formulae [4–6]. Sometimes, a suitable change of variable provides us with JWKB-like formulae solving the problem of energy spectra even exactly [4]. No doubts, however, that the latter possibility depends totally on a potential considered and a changing variable plays in such cases only an auxiliary role [11].

A change of variables is also an essential ingredient of a more general approach to the semiclassical approximations formulated by Maslov and his collaborators [13]. In the context of the latter approach the change-of-variable procedure is an inherent part of the continuation procedure of semiclassical series defined originally in some domain of the configuration space to another domain of the space. The relevant variable transformations used in the Maslov method are the canonical ones (in the sense of classical mechanics).

To establish clearly the relations as well as differences between the Maslov approach and the consideration performed in this paper, the former is discussed in some detail in appendix A. It is argued there that using fundamental solutions as we do in our paper is equivalent to the

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0305-4470/99/060955+22$19.50

method of Maslov *et al* in the semiclassical regime of the considered 1D problems but has many advantages over the Maslov procedure in the remainder of our investigations. In particular, the problem of Borel resummation central for our paper cannot be put and considered properly ignoring the existence of the fundamental solutions and their properties. After all, the method of Maslov *et al* is purely asymptotic from the very begining and any problem of resummation of the semiclassical series used in the method has not been considered as yet.

In particular, from the discussion of appendix A it follows clearly that the change-of-variable procedure considered in our paper does not have much to do with the canonical transformations used as a change of variable in the Maslov asymptotic method. The latter are the global Fourier transformations of wavefunctions whilst the former are the local point ones.

An improvement of the standard JWKB formulae achieved by the changing variable procedure appears as corrections typically having the form of an additional $\hbar$-dependent term in emerging effective potentials [1, 2, 4–6]. Since in all these cases of changing variable the standard JWKB formulae can be easily restored simply by $\hbar$-expansions of the improved ones, the latter seems to be some kind of hidden resummation of a partial (in the case of improvements only) or a full (when exact formulae emerge) standard semiclassical expansion corresponding to considered cases.

It is the aim of this paper to show that indeed this hidden resummation mentioned above really takes place whenever the change-of-variable procedure is applied to so-called fundamental solutions to the 1D Schrödinger equation [8,9] and that the effect of such a change of variable is equivalent to the Borel resummation of standard semiclassical expansions of the fundamental solutions multiplied by appropriately chosen $\hbar$-dependent constants.

A discussion of the fundamental solutions in the above context is essential since these solutions are the only ones with the property of being Borel summable [7]. Despite their rareness, the fundamental solutions when collected into a full set allow us to solve any 1D problem [8, 9] (see also a discussion below).

Because of the way in which a change of variable affects the fundamental solutions, a class of potentials which seems to be most natural for the corresponding discussing is the class of meromorphic potentials. This choice also enforces a suitable limitation of a class of variable transformations.

This paper is organized as follows. In the next section the fundamental solutions and their use are recalled. It is shown that there is a functional freedom in defining the solutions not affecting their Dirac form. In section 3 the analytic properties of the fundamental solutions and their dependence on the functional freedom mentioned above are discussed. In section 4 the standard semiclassical expansions of fundamental solutions and their properties, including their Borel summability, are reconsidered. It is shown that the functional freedom of the fundamental solutions is a freedom of multiplying them by suitably chosen $\hbar$-dependent constants holomorphic at $\hbar = 0$ and next restoring this functional freedom by the Borel resummation. In section 5 a class of variable transformations is discussed. It is shown that the net result of these transformations is to reproduce the functional freedom of the fundamental solutions discussed in the previous section. The Borel resummation aspects of a change-of-variable operation is also examined in this section. In section 6 the impossibility of achieving the exact JWKB formulae by a change-of-variable operation itself is shown. We conclude with section 7.

2. Stokes graphs and solutions they define

A standard way to introduce the fundamental solutions is a construction of a Stokes graph [7–9] for a given meromorphic potential $V(x)$ (the latter is assumed to be independent of the Planck constant $\overline{h}$). Such a construction, according to Fröman and Fröman [3] and Fedoriuk [17], can be performed in the following way.

Let Z denote a set of all the points of the x-plane at which $V(x)$ has its single and double poles. Let $f(x)$ be some other meromorphic function of x with none of its double pole being contained in Z. Let $\delta(x)$ be a meromorphic function, the unique singularities of which are double poles at the points collected by Z with the coefficients at all the poles equal to $\frac{1}{4}$ each. The latter function can be constructed in general with the help of the Mittag–Leffler theorem [18]. But in the case of a finite number of points contained in Z, $\delta(x)$ is simply a finite sum of double poles (with the coefficients equal to $\frac{1}{4}$ each) over all the points of Z.

Consider now a function

$$\tilde{q}(x,\overline{h}^2) = V(x) + \overline{h}^2 f(x) + \overline{h}^2\delta(x) - E \tag{1}$$

with E as the energy parameter which is assumed to be real.

It follows from the above assumptions that among all singularities of $\tilde{q}(x,\overline{h}^2)$ the simple poles can be provided by $f(x)$ only.

The presence and role of the δ-term in (1) are explained below. This term contributes to (1) if and only when the corresponding potential $V(x)$ contains simple or second-order poles. (Otherwise the corresponding δ-term is put to zero). We shall call it the Langer term.

The Stokes graph corresponding to the function $\tilde{q}(x,\overline{h}^2)$ consists now of Stokes lines emerging from roots (turning points) or simple poles of $\tilde{q}(x,\overline{h}^2)$. The points of the Stokes lines satisfy one of the following equations:

$$\mathcal{R}\int_{x_i}^{x}\sqrt{\tilde{q}(x,\overline{h}^2)}\,\mathrm{d}y = 0 \tag{2}$$

with x_i being a root or a simple pole of $\tilde{q}(x,\overline{h}^2)$.

The Stokes lines which are not closed end at these points of the x-plane (i.e. have the latter points as the boundaries) for which an action integral in (2) becomes infinite. Of course such points are singular for $\tilde{q}(x,\overline{h}^2)$ and can be its finite poles or its poles lying at infinity.

Each such singularity x_0 of $\tilde{q}(x,\overline{h}^2)$ defines a domain called a sector. This is the connected domain of the x-plane bounded by the Stokes lines and x_0 itself, with the latter point also being a boundary for the Stokes lines or being an isolated boundary point of the sector (as it is in the case of the second-order pole).

In each sector the LHS in (2) is only positive or negative.

Consider now the Schrödinger equation:

$$\Psi''(x) - \overline{h}^{-2}q(x)\Psi(x) = 0 \tag{3}$$

where $q(x) = V(x) - E$ (we have put the mass m in (3) to be equal to $\frac{1}{2}$).

Let us assume that a singularity x_0 of $\tilde{q}(x,\overline{h}^2)$ does not coincide with any of a second-order pole of $f(x)$. Then, following Fröman and Fröman (see also appendix B), one can define in each sector S_k having a singular point x_0 at its boundary a solution of the form:

$$\Psi_k(x) = \tilde{q}^{-\frac{1}{4}}(x)\cdot \mathrm{e}^{\frac{\sigma}{\overline{h}}W(x)}\cdot\chi_k(x) \qquad k = 1, 2, \ldots \tag{4}$$

where:

$$\chi_k(x) = 1 + \sum_{n\geqslant 1}\left(-\frac{\sigma\overline{h}}{2}\right)^n \int_{x_0}^{x}\mathrm{d}\xi_1\int_{x_0}^{\xi_1}\mathrm{d}\xi_2\ldots\int_{x_0}^{\xi_{n-1}}\mathrm{d}\xi_n\,\omega(\xi_1)\omega(\xi_2)\ldots\omega(\xi_n)$$
$$\times\left(1-\mathrm{e}^{-\frac{2\sigma}{\overline{h}}(W(x)-W(\xi_1))}\right)\left(1-\mathrm{e}^{-\frac{2\sigma}{\overline{h}}(W(\xi_1)-W(\xi_2))}\right)\cdots\left(1-\mathrm{e}^{-\frac{2\sigma}{\overline{h}}(W(\xi_{n-1})-W(\xi_n))}\right) \tag{5}$$

with

$$\omega(x) = \frac{\delta(x) + f(x)}{\tilde{q}^{\frac{1}{2}}(x,\overline{h}^2)} - \frac{1}{4}\frac{\tilde{q}''(x,\overline{h}^2)}{\tilde{q}^{\frac{3}{2}}(x,\overline{h}^2)} + \frac{5}{16}\frac{\tilde{q}'^2(x,\overline{h}^2)}{\tilde{q}^{\frac{5}{2}}(x,\overline{h}^2)} \tag{6}$$

and

$$W(x) = \int_{x_i}^{x} \sqrt{\tilde{q}(\xi,\overline{h}^2)}\,\mathrm{d}\xi \tag{7}$$

where x_i is a root of $\tilde{q}(x,\overline{h}^2)$ lying at the boundary of S_k.

In (4) and (5) a sign $\sigma\ (=\pm 1)$ and an integration path are chosen in such a way to have:

$$\sigma\mathcal{R}(W(\xi_j) - W(\xi_{j+1})) \leqslant 0 \tag{8}$$

for any ordered pair of integration variables (with $\xi_0 = x$). Such a path of integration is then called canonical.

The Langer δ-term appearing in (6) and in (7) is necessary to ensure all the integrals in (5) converge when x_0 is a first- or a second-order pole of $\tilde{q}(x,\overline{h}^2)$ or when solutions (4) are to be continued to such poles. It follows from (6) that each such a pole x_0 demands a contribution to $\delta(x)$ of the form $(2(x-x_0))^{-2}$, as has been already assumed in the corresponding construction of $\delta(x)$. We have also noticed earlier that the latter construction can be performed in general with the help of the Mittag–Leffler theorem. In practice, however, $\delta(x)$ can be guessed very frequently once the explicit form of $\tilde{q}(x,\overline{h}^2)$ is known (see [11] and appendix D, for example).

The constructions (4)–(7) have been performed by Fröman and Fröman [3] with the help of the change-of-variable procedure when the contributions of $\delta(x)$ and $f(x)$ to $\tilde{q}(x,\overline{h}^2)$ emerge as a result of this procedure. This construction suggests, however, that there must be room to introduce to solution (4) a sum $\delta(x) + f(x)$ as its free parameter independent of any change-of-variable procedure. In appendix B we show that indeed this is the case and the constructions of Fröman and Fröman as well as the ones in (4)–(7) simply make use of this freedom.

3. Fundamental solutions and their properties as functionals of $V(x)$ and $f(x)$

It is clear that the poles of $V(x)$ and $f(x)$ are also the singular points for each of the three factors of the representation (4) for the solutions $\Psi_k(x,\overline{h})$. However, since the latter satisfy the Schrödinger equation (3) only the singularities of the potential $V(x)$ can affect (and they do!) the holomorphicity of $\Psi_k(x,\overline{h})$ at these points, i.e., at the singular points of $f(x)$ which do not coincide with those of $V(x)$ all $\Psi_k(x,\overline{h})$ have to be analytical. But, by their construction, $\Psi_k(x,\overline{h})$ which have been defined at the latter points vanish at these points together with their first derivatives. It means that all these $\Psi_k(x,\overline{h})$ have to vanish identically. (In fact it is the third factor of these solutions in (4) which vanishes identically).

An important conclusion which follows from the above discussion is that the constructions of $\Psi_k(x,\overline{h})$ in the sectors corresponding to these singular points of $f(x)$ which do not coincide with those of $V(x)$ can be neglected, i.e., we are left only with the solutions generated by the sectors defined by the genuine singularities of these solutions coinciding with those of $V(x)$.

Our initial assumption that x_0 differs from any double pole of $f(x)$ is naturally related to the fact that in general the solutions (4)–(7) cannot be constructed at such poles even formally because the integrals in RHS of (5) are divergent at these poles. Contrary to the second-order poles of $V(x)$, the latter divergences cannot be compensated, as is seen from (6), by the appropriate Langer term different from $f(x)$ itself. If, however, the latter possibility happens then the coefficient at each double pole of $f(x)$ would have to be equal to $\frac{1}{4}$. But then the fundamental solutions constructed at these poles again have to vanish identically for the reasons discussed above.

Among all the singularities of $V(x)$ there are simple poles which have to be distinguished in our discussion. This is because the sectors determined by these singularities are $\overline{h}$-dependent (this dependence is introduced by the Langer term) and shrink to points when $\overline{h} \to 0$. (The limit points are of course the actual positions of the simple poles themselves as the singularities of $V(x)$). Because of that the solutions (4)–(7) constructed in these sectors differ from the remaining ones in their properties (they are not Borel summable, for example, see appendix C) and therefore only the latter will be investigated further.

Therefore, we consider as fundamental solutions $\Psi_k(x, \overline{h})$ only those defined by (4)–(7) which are constructed at all the poles of $V(x)$ (including that at infinity) except its simple ones.

The following are the properties of the fundamental solutions.

(1) In a domain D_k of the x-plane where condition (8) is satisfied (the so-called canonical domain) the series in (5) defining χ_k is uniformly convergent. χ_k itself satisfies the following initial conditions:

$$\chi_k(x_0) = 1 \qquad \text{and} \qquad \chi_k'(x_0) = 0 \tag{9}$$

corresponding to the equation:

$$\chi_k(x) = 1 - \frac{\sigma\overline{h}}{2}\int_{x_0}^{x} \mathrm{d}y\, \omega(y)\chi_k(y) - \frac{\sigma\overline{h}}{2}\tilde{q}^{-\frac{1}{2}}(x)\chi_k'(x) \tag{10}$$

this function has to obey as a consequence of the Schrödinger equation (3) and the initial conditions (9).

(2) The fundamental solution $\Psi_k(x, \overline{h})$ defined in the sector S_k and continued to the canonical domain D_k $(S_k \subset D_k)$ has there the following two basic properties:

(a) It can be expanded in D_k into a standard semiclassical series obtained by iterating equation (10) and taking into account the initial conditions (9);

(b) The emerging semiclassical series is Borel summable in S_k to the solution itself.

(3) The fundamental solutions are pairwise independent and collected into a full set they allow one to solve *any* 1D problem.

Let us note also that the fundamental solutions seem to be distinguished by the property (b) above, i.e., they seem to be the unique solutions to the Schrödinger equation (3) with this property at least for the polynomial potentials [7]. The property (b) has been proven earlier for the polynomial potentials [8] and for the meromorphic ones it is proven in appendix C.

4. Standard semiclassical expansions and Borel summability of fundamental solutions

By a standard semiclassical expansion for χ we mean the following series:

$$\begin{aligned}
&\chi(x) \sim C(\overline{h}) \sum_{n\geqslant 0}\left(-\frac{\sigma\overline{h}}{2}\right)^n \kappa_n(x) \\
&\kappa_0(x) = 1 \\
&\kappa_n(x) = \int_{x_0}^{x} \mathrm{d}\xi_n\, \tilde{D}(\xi_n)\int_{x_0}^{\xi_n} \mathrm{d}\xi_{n-1}\, \tilde{D}(\xi_{n-1})\ldots\int_{x_0}^{\xi_3} \mathrm{d}\xi_2\, \tilde{D}(\xi_2)\int_{x_0}^{\xi_2} \mathrm{d}\xi_1\, (\tilde{q}^{-\frac{1}{4}}(\xi_1)(\tilde{q}^{-\frac{1}{4}}(\xi_1))'' \\
&\qquad\qquad + \tilde{q}^{-\frac{1}{2}}(\xi_1)(\delta(\xi_1) + f(\xi_1))) \qquad n = 1, 2, \ldots
\end{aligned} \tag{11}$$

with

$$\tilde{D}(x) = \tilde{q}^{-\frac{1}{4}}(x, \overline{h}^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\tilde{q}^{-\frac{1}{4}}(x, \overline{h}^2) + \tilde{q}^{-\frac{1}{2}}(x, \overline{h}^2)(\delta(x) + f(x)) \tag{12}$$

where

$$C(\overline{h}) = \sum_{n\geqslant 0} C_n \left(-\frac{\sigma\overline{h}}{2}\right)^n \tag{13}$$

and where the choices of a point x_0 and constants C_k, $k = 1, 2, \ldots$, are arbitrary. However, for the particular χ_k (as defined by (5), for example) these choices are of course definite (if x_0 is given by the lower limit of the integrations in the expansion (5) then $C(\overline{h}) \equiv 1$). Nevertheless, even in such cases the choice of x_0 can be arbitrary whilst the emerging corresponding constants C_k depend somehow on x_0 [7].

The representation (11) is standard in a sense that any other one can be brought to (11) by redefinitions of the constants C_k. Therefore, any semiclassical expansion can be uniquely given by fixing x_0 and the constants C_k. Conversly, multiplying a given semiclassical expansion by an asymptotic series as defined by the last series in (11) with other constants C_k, $k = 1, 2, \ldots$, one can obtain any other semiclassical expansion.

As we have mentioned above, the semiclassical series for χ is Borel summable for x staying in the sector S_k where χ is defined. In fact this summability takes place not only inside a circle $\mathcal{R}(\overline{h}^{-1})^* = (2R)^{-1}$ of the $\overline{h}$-plane demanded by the conditions of the Watson–Sokal–Nevanlinna theorem [10] but rather in the circle $|\overline{h}| < \overline{h}_0$ of the $\overline{h}$-plane cut along the radius $|\arg\overline{h}| = \pm\pi$.

Let us now make a crucial observation that a solution $\Psi_k(x, \overline{h})$ constructed at the singular point x_0 for a given function $f(x) \neq 0$ and a solution $\tilde{\Psi}_k(x, \overline{h})$ constructed at the same point x_0 but corresponding to the choice $f(x) \equiv 0$ both satisfy the Schrödinger equation (3) with the same potential $V(x)$ and therefore have to coincide with each other up to some multiplicative $\overline{h}$-dependent constant $C_k(\overline{h})$ (see appendix B for a proof of this statement), i.e.:

$$\Psi_k(x) = C_k(\overline{h})\tilde{\Psi}_k(x) \qquad k = 1, 2, \ldots \tag{14}$$

with $C_k(\overline{h})$ given by

$$C_k(\overline{h}) = \exp\left[\sigma\overline{h}\int_{x_i}^{x_0} \frac{f(x)}{\sqrt{q(x)+\overline{h}^2\delta(x)} + \sqrt{q(x)+\overline{h}^2\delta(x)+\overline{h}^2 f(x)}}\,\mathrm{d}x\right] \tag{15}$$

where the coefficient $C_k(\overline{h})$ was calculated by taking a limit $x \to x_0$ on both the sides of (14).

It follows from (15) that $C_k(\overline{h})$ is *holomorphic* at $\overline{h} = 0$ and because of that its Taylor series at this point is trivially Borel summable.

The equality (14) proves therefore that the solution on its LHS can be obtained by multiplying the solution on its RHS by the constant $C_k(\overline{h})$, then making the semiclassical expansion of this product and next summing again *á la* Borel the resulting semiclassical series.

Our main conclusion, however, which follows from the discussion of the previous section and from the last reasoning as well is that the generality of the constructions of fundamental solutions containing an arbitrary function $f(x)$ is only apparent, i.e., the most 'economical' forms of these solutions and of the Stokes graphs corresponding to them are those obtained for $f(x) \equiv 0$. Namely, the corresponding Stokes graphs are then composed from the minimal number of Stokes lines and sectors.

Nevertheless, introducing the function $f(x)$ to the fundamental solutions is not completely useless and it can help in some situations [11]. Some advantages which follows from using the f-function depending on $\overline{h}$ are also discussed in section 6.

5. Change of variable as Borel resummations

As he have mentioned earlier, the constructions (4)–(7) of the fundamental solutions were originally performed by Fröman and Fröman using the change-of-variable procedure in the

Schrödinger equation (3). Namely, putting $x = x(y)$ we keep the form of the Schrödinger equation (3) unchanged if simultaneously we make a substitution: $\Phi(y) \equiv x'^{-\frac{1}{2}}(y)\Psi(x(y))$, so that $Q(y)$ corresponding to $\Phi(y)$ in its Schrödinger-like equation is given by:

$$Q(y) = q(x(y))x'^2(y) - \frac{1}{2}\hbar^2\left[\frac{x'''(y)}{x'(y)} - \frac{3}{2}\frac{x''^2(y)}{x'^2(y)}\right]$$
$$= q(x)\frac{1}{y'^2(x)} + \frac{1}{2}\hbar^2\left[\frac{x'''(y)}{x'(y)} - \frac{3}{2}\frac{x''^2(y)}{x'^2(y)}\right]\frac{1}{y'^2(x)}\bigg|_{x=x(y)} \tag{16}$$

where $y(x)$ is a function inverse to $x(y)$. The expressions in the parentheses in (16) are called the Schwarzians (of $x(y)$ and of $y(x)$, respectively).

It follows from (16) that to have $Q(y)$ meromorphic in the y-plane we have to assume also some analytic properties of $x(y)$. The meromorphicity of $x(y)$ would be enough for this goal but it could limit excessively a range of allowed $x(y)$ (see appendix D). Therefore we assume the allowed family of $x(y)$ to depend on $q(x)$ in a sense that the family is a collection of all the complex functions of y which provides us with both $q(x(y))$ and $x'^2(y)$ as meromorphic functions of y. It is then easy to check that this assumption is sufficient not only for $Q(y)$ to be a meromorphic function of y but also for the Schwarzian of $y(x)$ to be a meromorphic function of x. (The last property of this Schwarzian is important when we come back both to the initial variable x and the initial wavefunction $\Psi(x)$).

Let us characterize a possible singularity structure of the inverse function $y(x)$ and its derivative.

Since $x'^2(y)$ is meromorphic then the possible singularities of $y'(x)$ ($\equiv 1/x'(y)$) at finite points $y_0 = y(x_0)$ are branch points of the type $(x - x_0)^{-n/(n+2)}$, $n \geqslant 1$, (where $x'^2(y)$ has an nth order root). It is, however, easy to check that when x_0 coincides with some pole of $q(x)$ then to keep $q(x(y))$ meromorphic at y_0 we have to assume n to be even. Therefore, it is rather $x'(y)$ itself (than its square) which is then holomorphic at y_0.

If $x'^2(y)$ vanishes when y escapes to infinity (in some direction on the y-plane) so that $x(\infty e^{i\phi}) = x_0$, where x_0 can be finite or infinite, we shall *assume* that $y'^2(x)$ approaches infinity according to the rules: $y'^2(x) \sim (x - x_0)^{-2\alpha}$, for finite x_0, or $y'^2(x) \sim x^{\alpha}$ for the infinite one, with positive $\alpha \geqslant \frac{1}{2}$ in the first case and with $\alpha > 0$ in the second one. The last assumptions are in accordance with the meromorphicity of the Schwarzian derivative of $y(x)$. Of course, the last assumptions still limit somehow a family of the functions $x(y)$ which can be used in the change-of-variable procedure.

Now, we can apply the procedure of the previous section to construct the fundamental solutions for $\Phi(y)$ choosing, however, their most 'economical' form (i.e. putting $f(x) \equiv 0$ in (1)). Of course we need not do it in all, but only in those singular points of $Q(y)$ which are images of the corresponding singularities of $q(x)$ since we want to come back to the original wavefunctions $\Psi_k(x)$.

Let us review therefore relevant possibilities.

If for some finite singularity x_0 of $q(x)$ (being neither its simple nor a double pole; see, however, a further discussion) its image $y_0 = y(x_0)$ is finite then the corresponding singularity of $Q(y)$ at y_0 is again a pole of the same or higher order (depending on the lowest order of the derivative of $x(y)$ which does not vanish at y_0).

A simple pole of $q(x)$ with a finite image survives as a simple one for $Q(y)$ if $x'^2(y)$ does not vanish at this pole. Otherwise $Q(y)$ gains a double pole at the corresponding point from the Schwarzian of $x(y)$.

A double pole of $q(x)$ in the case considered always survives as a double one for $Q(y)$ no matter what the order of the root of $x'^2(y)$ at the corresponding point is.

If the image of a finite singularity x_0 of $q(x)$ lies, however, at the infinity of the y-plane then it follows from our assumption that y approaches the infinity as $(x - x_0)^{-\alpha+1}$, for $\alpha > \frac{1}{2}$, or as $\log(x - x_0)$. In the first case it can happen, however, that at the infinite image of x_0 $Q(y)$ vanishes sufficiently quickly so that the corresponding action defined by it is finite at the infinity. This takes place for example when the order of the pole of $q(x)$ at x_0 is less than 2α. Therefore in such a case a corresponding fundamental solution cannot be defined at this infinite point.

If x_0 is an infinite singular point of $q(x)$ with its image being also infinite point of the y-plane then, according to our assumption, the latter infinity is a singular point of $Q(y)$ if $q(x)$ diverges faster than some power of x.

Taking all these into account we see that at each double and higher-order pole of $q(x)$ the image of which survives as a singular point of $Q(y)$ we can construct the following fundamental solution to the Schrödinger equation (3):

$$\tilde{\Psi}_k = (y'^2 \tilde{Q}(y(x)))^{-\frac{1}{4}} \exp\left[\frac{\sigma}{\hbar}\int_{x_i}^{x}\sqrt{y'^2(\xi)\tilde{Q}(y(\xi))}\,\mathrm{d}\xi\right]\tilde{\chi}_k(y(x)) \qquad k = 1, 2, \ldots \tag{17}$$

where $\tilde{\chi}_k(y)$ is constructed according to (5)–(7) by making there substitutions: $x \to y (= y(x))$, $\delta(x) \to \tilde{\delta}(y)$, $\tilde{q}(x) \to \tilde{Q}(y)$, $\omega(x) \to \tilde{\omega}(y)$, $W(x) \to \tilde{W}(y)$ and $x_0 \to y_0 (= y(x_0))$. The corresponding $\tilde{\delta}(y)$-term can be constructed only from those double poles of $Q(y)$ which are finite images of the corresponding double poles of $q(x)$, as is assumed from now on.

The form of the solution (17) shows that the change of variable leads us again to some fundamental solution i.e. it keeps the form (4) of these solutions.

Comparing the constructions (4)–(7) with (16) we can make the following identification:

$$\delta(x) + f(x) = \tilde{\delta}(y(x))y'^2(x) - \frac{3}{4}\frac{y''^2(x)}{y'^2(x)} + \frac{1}{2}\frac{y'''(x)}{y'(x)}. \tag{18}$$

It follows from (18) that to have its RHS as a holomorphic function of x we should have to assume additionally the meromorphicity of $\tilde{\delta}(y(x))$ as a function of x. This assumption restricts still more the allowed family of $y(x)$. However, we shall see below that the limited role of the change-of-variable procedure when applied to the fundamental solutions will allow us to forget about the $\tilde{\delta}(y)$-term in these cases, so that the assumptions made by us earlier about the transformations $x(y)$ are still sufficient.

We show below that under the assumptions we have made in this section the LHS of (18) is reconstructed by its RHS.

First, let us note that the Schwarzian term in (18) generates double poles in all the finite points of the x-plane where $y'(x)$ is singular. The coefficients at these double poles are determined by the power indeces of these singularities. As a rule the values of these coeficients are not equal to $\frac{1}{4}$. Note, however, that the double poles generated in this way can be ignored if they do not coincide with the singularities of $q(x)$.

In the opposite case, however, when some of the double poles of the Schwarzian coincide with some of the singularities of $q(x)$, we have to check whether they contribute to the RHS according the rule described in section 2. Let us consider, therefore, the corresponding possibilities listed earlier.

(1) x_0 is a double pole of $q(x)$ and $y_0 = y(x_0)$ is finite. Then y_0 is also a double pole for $Q(y)$ and is therefore corrected by the Langer term $\tilde{\delta}(y)$. It is then easy to check that this term (multiplied by $y'^2(x)$) together with the Schwarzian contributes exactly the term $(2(x - x_0))^{-2}$ to the RHS of (18);

(2) x_0 is a double pole of $q(x)$ with the infinite image. Then this image is, of course, not represented in the Langer term $\tilde{\delta}(y)$. This infinity is then a singular point of $Q(y)$ if y

runs to it as $\log(x - x_0)$. To the RHS of (18) contributes the Schwarzian only and one can check that it is again the term $(2(x - x_0))^{-2}$;

(3) x_0 is a higher (than the second) order pole of $q(x)$ with the finite image. Then only the Schwarzian contributes to the RHS of (18) with its appropriate second-order pole so that the corresponding structure of the LHS at this pole is kept;

(4) x_0 is a higher-order pole with the infinite image. Then $Q(y)$ vanishes at this image and the corresponding action is finite—there is no a fundamental solution $\Phi(y)$ to be constructed at this infinite point;

(5) x_0 is an infinite singular point of $q(x)$ and is transformed by $y(x)$ into an infinite singular point of $Q(y)$. It is then easy to check that the Schwarzian does not contribute at x_0 (it vanishes) and the point remains a singular one at least for $q(x)$. Therefore the singularity structure of the LHS of (18) is maintained.

The possibilities (1)–(5) above exhaust all the relevant cases and prove in this way our statement that the RHS of (18) has the same singularity structure as its LHS up to these singularities of $q(x)$ which are transformed into regular points of $Q(y)$ and its actions.

It is worth noting here that, as follows from the case (2) above, a part of the Langer term on the LHS of (18) is reconstructed by the Schwarzian on the RHS so that the equality (18) cannot be split into two others, for the corresponding deltas and the rest.

Thus we have shown that choosing $y(x)$ properly we can recover by the change-of-variable procedure the general structure of the fundamental solutions described by (4)–(7).

Nevertheless, it seems to be obvious that the direct constructions of the Langer term $\delta(x)$ as well as the corresponding function $f(x)$ (if necessary for some reasons) is much simpler than looking for the corresponding $y(x)$ by constructing $x(y)$ (as a meromorphic function) and trying to invert the latter.

Therefore a question arises about an advantage of making a change of variable in the context of looking for the fundamental solutions. If we assume that we can put $f(x)$ equal to zero, then such an advantage can be connected with the possibility of constructing the Langer term in a way different than the one relied on the Mittag–Leffler theorem.

Namely, we can try instead to construct a $y = y(x)$ with the property to escape logarithmically to infinity each time when x approaches a position of some double pole of $q(x)$. For this goal we can put $y = \log z(x)$ and construct $z(x)$ as a holomorphic function of x having its unique simple roots at all the double poles of $q(x)$. The latter construction can be performed using the Weierstrass product theorem [19]. Then the Schwarzian of $y(x)$ provides us with the desired Langer term. By this construction, however, the appearance of the residual f-function is in general unavoidable. But if we want to keep only the emerging Langer δ-term we can of course discard the residual f-function. A simple example of such a construction is given in appendix D.

Finally, needless to say, it follows that the fundamental solutions obtained with the help of the change-of-variable procedure described above are Borel summable and, as follows from section 4, can be obtained as the Borel resummation of the other ones multiplied by suitably chosen $\hbar$-dependent constants.

Of course, it seems to be obvious that it is not a good idea to get the effect of changing variable in the latter way. Instead we can consider the above relation of the variable transformation to the Borel resummation of the fundamental solutions, i.e. to the operation which is intimately connected with the semiclassical expansions, as an explanation of why it could improve many semiclassical approximations. The well known Langer substitution in the case of the Coulomb potential serves here as the most famous example [1, 2]. Note, however, that the *exactness* of the corresponding JWKB formula for the Coulomb case is *not* a direct consequence of this substitution but rather a result of a hidden symmetry of the Coulomb potential displayed after another change of variable in the Schrödinger equation with this potential [11].

6. Change of variable and exactness of JWKB quantization formulae

The above-mentioned correcting properties of a change of variable in the Schrödinger equation can of course, be discussed equivalently in terms of the f-function introduced in section 2, which is even more convenient. Let us raise the question of how far the semiclassical results can be corrected by choosing $f(x)$ properly. By these corrections we mean a decreasing contribution to the semiclassical formulae of the third factor of the representation (4) of the fundamental solutions in favour of the first two.

It follows from (4)–(7) and (14) modelling $f(x)$ that we can change all the three factors. The question is therefore whether it is possible to choose $f(x)$ in such a way as to put the corresponding χ-factors equal to 1 (the latter number being the limiting form of these factors)?

To answer the question consider equation (14), substituting there the corresponding forms (3) of the wavefunctions $\tilde{\Psi}_k(x)$ and $\Psi_k(x)$. The discussion can be made more transparent by using the following exponential representation for $\tilde{\chi}_k(x,\overline{h})$ and $\chi_k(x,\overline{h})$ (the latter functions correspond to the wavefunctions $\tilde{\Psi}_k(x)$ and $\Psi_k(x)$, respectively):

$$\tilde{\chi}_k(x,\overline{h}) = \exp\left(\int_{x_0}^{x} \tilde{\rho}_k(\xi,\overline{h})\,\mathrm{d}\xi\right) \qquad \chi_k(x,\overline{h}) = \exp\left(\int_{x_0}^{x} \rho_k(\xi,\overline{h})\,\mathrm{d}\xi\right) \tag{19}$$

so that

$$\tilde{\rho}_k(x,\overline{h}) = \frac{\tilde{\chi}_k'(x,\overline{h})}{\tilde{\chi}_k(x,\overline{h})} \qquad \rho_k(x,\overline{h}) = \frac{\chi_k'(x,\overline{h})}{\chi_k(x,\overline{h})}. \tag{20}$$

Then the following relation comes out from the equality (14):

$$\begin{aligned}\rho_k(x,\overline{h}) = \tilde{\rho}_k(x,\overline{h}) + \sigma\overline{h}&\frac{f(x)}{\sqrt{\tilde{q}(x,\overline{h}^2)} + \sqrt{\tilde{q}(x,\overline{h}^2) + +\overline{h}^2 f(x)}}\\ &-\frac{\overline{h}^2}{4}\frac{\tilde{q}(x,\overline{h}^2)}{\tilde{q}(x,\overline{h}^2) + \overline{h}^2 f(x)}\left(\frac{f(x)}{\tilde{q}(x,\overline{h}^2)}\right)'.\end{aligned} \tag{21}$$

It follows from (20) that both $\tilde{\rho}_k(x,\overline{h})$ and $\rho_k(x,\overline{h})$ are Borel summable and from (21) that their Borel transforms differ by a function holomorphic on the whole Borel plane (since both the functions $f(x)$ and $\delta(x)$ are $\overline{h}$-independent). Therefore it is clear that one cannot find such $f(x)$) to cause $\rho_k(x,\overline{h})$ to disappear, i.e., one cannot be left in $\Psi_k(x)$ with its first two JWKB factors only. This is because $\tilde{\rho}_k(x,\overline{h})$ is singular at $\overline{h} = 0$.

However, making $f(x)$ also $\overline{h}$-dependent but choosing it holomorphic at $\overline{h} = 0$ we can achieve a result when the first n terms of the semiclassical expansion of $\rho_k(x,\overline{h})$ vanish. The latter is possible globally (i.e. independently of k) since the semiclassical expansions of $\rho_k(x,\overline{h})$ are k-independent (i.e. these expansions do not contain any integration on the x-plane, see for example [16]). One of our earlier paper is a good illustration of this possibility [6] (see also a comment below).

To achieve the goal of vanishing $\rho_k(x,\overline{h})$ we would have to use $f(x,\overline{h})$ as singular at $\overline{h} = 0$ and therefore expected to satisfy all the necessary conditions of the Watson–Sokal–Nevanlinna theorem to be Borel summable. In such a case, $f(x,\overline{h})$ becomes, similarly to $\rho_k(x,\overline{h})$, sector dependent i.e. within the class of the Borel summable functions there is no possibilty to define a *global* $y(x,\overline{h})$ which could provide us with $\Psi_k(x,\overline{h})$ deprived of its χ_k-factor for all k simultaneously. In a more obvious way one can conclude this from (20) putting there $\rho_k(x,\overline{h})$

equal to zero and then treating the equation obtained in this way as the differential one for $f(x,\hbar)$ where $\tilde{\rho}_k(x,\hbar)$ is assumed to be given. However, for any two different k there are two different $\tilde{\rho}_k(x,\hbar)$ and in consequence two different solutions for $f(x,\hbar)$ have to emerge.

Needless to say, the above negative result for the existence of the corresponding function $f(x,\hbar)$ is also negative for the respective $y(x,\hbar)$ which can be obtained as a solution to equation (18) and which changes the variable in the Schrödinger equation (3).

Because of (14), the last result can also be expressed in terms of the $\hbar$-dependent constants present in this relation. Namely, it follows from the above discussion that there is no choice of the constants C_k which could cause all the corresponding χ_k to be reduced to unity if all the constants as given by (15) are to be defined only by one global $f(x,\hbar)$ (or, alternatively, by some $y(x,\hbar)$ realizing the underlying change of the x-variable).

Nevertheless, it is always possible to make such a global choice of $f(x,\hbar)$ (or equivalently $y(x,\hbar)$ or the constants C_k) to produce simultaneously all the fundamental solutions for which the series in (5) start with an arbitrary high power of $\hbar$. Of course, this is the case of $f(x,\hbar)$ holomorphic at $\hbar = 0$. Such a choice corresponds to a total effect of repeating changes of variable when for each subsequent Schrödinger-like equation a new independent variable is the action i.e. $y'^2(x) = q(x)$. The 'lacking' powers of $\hbar$ are collected then in $(y'^2(x,\hbar)\tilde{Q}(x,\hbar))^{-\frac{1}{4}}$ and in the corresponding exponential factors of the solutions (4). These two factors are then the sources of new JWKB approximations generalizing the conventional ones [6].

The impossibility to reduce the form (4) of the fundamental solutions to their first two factors by choosing properly some $f(x,\hbar)$ (the same for all the solutions) has an immediate consequence for obtaining the exact JWKB formulae for quantization of energy levels. Namely, quantizing the levels in a potential well it is impossible, in general, to get the quantization formula in the pure JWKB form:

$$\exp\left[\frac{\sigma}{\hbar}\oint_K \sqrt{q(x) + \hbar^2\delta(x) + \hbar^2 f(x,\hbar)}\,\mathrm{d}x\right] = -1 \tag{22}$$

by choosing $f(x,\hbar)$ properly. The cases where this is possible are rare and are related rather to some particular symmetries of potentials considered [11].

To show this let us quantize a 1D quantum system with the help of the fundamental solutions (it has been described in many of our earlier papers [1, 13, 14, 16]). Such a system can be defined by describing positions and orders of poles and roots of $\tilde{q}(x,\hbar^2)$ but instead of taking some particular example of the latter we prefer to define $\tilde{q}(x,\hbar^2)$ by such a description stressing in this way its arbitrariness as well as the potential generality of arguments.

For simplicity we shall assume, however, that after a change of variable the effective potential which emerges has only one absolute minimum so that we can choose the quantized energy level to lie in the well with this minimum and below the rest. Then, there are only two real turning points x_1, x_2 of $\tilde{q}(x,\hbar^2)$ whilst the rest of them are complex and conjugated pairwise (we assume $\tilde{q}(x,\hbar^2)$ and E to be real). We assume also that the problem has been limited to a segment $z_1 \leqslant x \leqslant z_2$ at the ends of which $\tilde{q}(x,\hbar^2)$ has poles. But if we wish we can push any of $z_{1,2}$ (or both of them) to $\mp\infty$ respectively.

To write the corresponding quantization condition for energy E and to handle simultaneously the cases of second- and higher-order poles we assume z_1 to be the second-order pole and z_2 to be the higher ones.

It is also necessary to fix to some extent the closest enviroment of the real axis of the x-plane to draw a piece of SG sufficient to write the quantization condition. To this end we assume x_3 and $\bar{x}_3$ as well as x_4 and $\bar{x}_4$ to be another four turning points and z_3 and $\bar{z}_3$ another two second-order poles of $\tilde{q}(x,\hbar^2)$ closest to the real axis. We assume also a possible divergence at infinity. Then a relevant part of the Stokes graph can look as in figure 1.

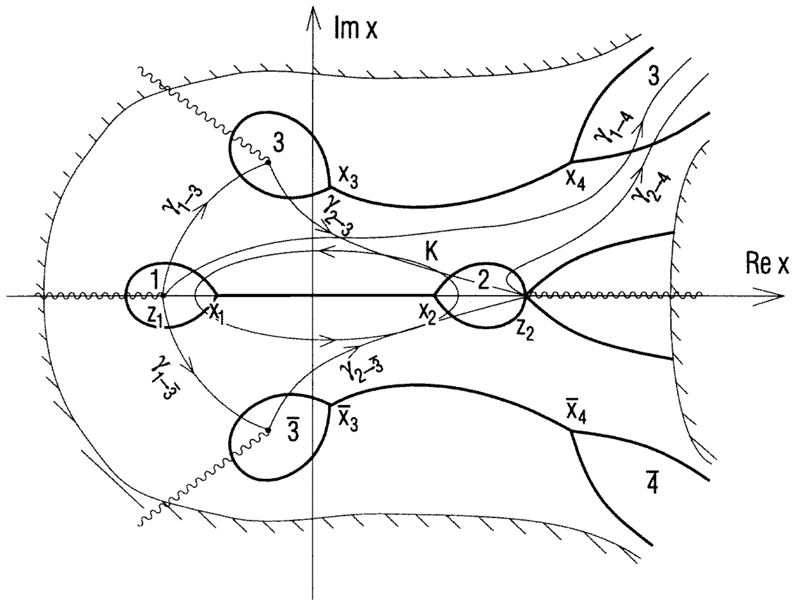

Figure 1. The SG corresponding to general quantization rule (23).

There is no unique way of writing the quantization condition corresponding to the figure. Some possible three forms of this condition can be written as [8]:

$$\exp\left[\frac{\sigma}{\bar{h}}\oint_K \sqrt{q(x)' + \bar{h}^2\delta(x) + \bar{h}^2 f(x,\bar{h})}\,\mathrm{d}x\right] = -\frac{\tilde{\chi}_{1\to3}(\bar{h})\tilde{\chi}_{2\to\bar{3}}(\bar{h})}{\tilde{\chi}_{1\to\bar{3}}(\bar{h})\tilde{\chi}_{2\to3}(\bar{h})} = -\frac{\tilde{\chi}_{1\to4}(\bar{h})\tilde{\chi}_{2\to\bar{3}}(\bar{h})}{\tilde{\chi}_{1\to\bar{3}}(\bar{h})\tilde{\chi}_{2\to4}(\bar{h})} \tag{23}$$

where $\tilde{\chi}_{k\to j}(\bar{h}), k, j = 1, 2, 3, 4$ are calculated for $x \to z_j$ with the help of formula (5), i.e. $\tilde{\chi}_{k\to j}(\bar{h}) = \lim_{x\to z_j} \tilde{\chi}_k(\bar{h})$ along a canonical path. The closed integration path K is shown in figure 1. In the figure the paths $\gamma_{1\to3}$, $\gamma_{2\to3}$, etc are the integration paths in formula (5) between the singular points z_1 and z_3, z_2 and z_3, etc respectively, whilst the wavy lines designate corresponding cuts of the x-Riemann surface on which all the fundamental solutions are defined.

The condition (21) is *exact*. Its LHS has just the JWKB form. If we substitute each $\tilde{\chi}_{k\to j}(\bar{h})$ in (23) by unity (which these coefficients approach when $\bar{h} \to 0$) we obtain the well known JWKB quantization rule (22) which, in general, is only an approximation to (23).

Now, since there is no such a *global* $f(x,\bar{h})$ (alternatively, an x-variable transformation $y(x,\bar{h})$) by which all $\tilde{\chi}_{k\to j}(\bar{h})$ in (23) could become simultaneously equal to unity the RHS of (23) cannot be reduced to unity by any such $f(x,\bar{h})$ i.e. the JWKB formula provided in this way by (23) is always only an approximation. As we have mentioned, some additional symmetry conditions have to be satisfied by the initial $q(x)$ to provide us with such an exact JWKB formula [11].

7. Conclusions

In this paper we have shown that the Borel summable fundamental solutions to Schrödinger equation with the meromorphic potentials can be modified by an almost arbitrary meromorphic function and that this modification is equivalent to the appropriate Borel resummations of the unmodified solutions multiplied by properly chosen $\hbar$-dependent constants. We have also shown that such modifications are always equivalent to performing some change of variable in the corresponding Schrödinger equation (3) when the derivative of the substituted variable depends meromorphicaly on the new one.

We have also argued that the advantages of variable transformations when applied to the fundamental solutions are rather limited and seem to provide us in the best case with the proper form of the Langer terms.

The relation of the effect of a variable transformation in the fundamental solutions to suitable Borel resummations of these solutions multiplied earlier by respective $\hbar$-dependent constants explains a little about the mysterious improvement of the JWKB formulae obtained as a result of such a change of variable.

On the other hand, we have also provided arguments that no such change of variable can reduce an exact quantization formula to its pure JWKB form. The latter case can happen only due to particular symmetry properties of quantized potentials [11, 21].

Acknowledgments

We would like to thank the referees of our paper whose comments and advice helped us to significantly improve its quality. SG was supported by KBN 2PO3B 07610. PM was supported by the Łódź University Grant No 580.

Appendix A

The Maslov method is formulated for an arbitrary linear partial differential equation (LPDE) having as its semiclassical partner a dynamical system with a finite number of degrees of freedom [13]. Maslov's semiclassical theory of solutions to the corresponding LPDE is developed on the 'classical' objects known in the classical mechanics as Lagrangian manifolds [13, 14]. Limited to the one-degree-of-freedom case and to the 1D Schrödinger equation, the Lagrangian manifolds are nothing but the 1D classical trajectories in the corresponding 2D phase space. Exact solutions to the stationary Schrödinger equation having particular Dirac forms (4) can be naturally redefined to live on the Lagrangian manifold corresponding to a given energy. However, to cover by such a description the whole coordinate domain which the corresponding wavefunctions are defined on, the imaginary time evolution of the classical equations of motion has also to be switched on to take into account so-called 'classically forbidden regions'. The emerging Lagrangian manifold then contains branches corresponding to the real time motions (performed in classically allowed regions) as well as to the imaginary ones with the imaginary part of the momentum in the latter case playing the role of the classical momentum. Of course, the semiclassical conditions for the considered global wavefunction are the following: it has to vanish exponentially when $\hbar \to 0$ in the classically forbidden regions and to oscillate in the classically allowed ones.

Unfortunately, the Dirac representation of these solutions considered as functions of the coordinate cannot be defined globally on the above Lagrangian manifold, being singular at points where the manifold branches making a matching procedure of the solutions defined on different branches impossible. These singular points are, in general, called the caustic ones but in the 1D case they are known as turning points. Maslov and Fedoriuk's remedy to solve this arising 'connection problem' is to change the coordinate variable around such points into the corresponding momentum, i.e., to change the coordinate representation of the wavefunction into the momentum one preserving the Dirac form of the solution. Assuming the wavefunction to be normalized, its latter representation can be given formally by the Fourier transformation of the former. In the new representation the wavefunction is then regular at the coordinate turning points of Lagrangian manifold (being on the other hand singular at the emerging momentum turning points). The inverse Fourier transformation considered close to a coordinate turning point provides us again with the solution in the coordinate representation given on both the sides of the chosen coordinate turning point. As we have mentioned above, the semiclassical limit condition for the latter solution is of course to vanish exponentially (when $\hbar \to 0$) on one side of the turning point and to oscillate on the other. This condition determines the way the local solutions determined on both the sides of each turning point and having the Dirac form are to be matched.

The above idea of matching the solutions on different branches of the Lagrangian manifold does not seem to be effective for the exact solutions to Schrödinger equation but it becomes so when the solutions in their Dirac forms are substituted by their corresponding semiclassical series. This is, in fact, the subject of the original approach of Maslov and collaborators. Namely, in such a case the classically forbidden parts of the solutions disappeared completely (being exponentially small) and the remaining ones are then given uniquely on the classically allowed branches of the Lagrangian manifold. The matching procedure then connects only two oscillating solutions separated by the corresponding turning point. The underlying Fourier transformation then becomes effectively a point transformation determining the connection. As is well known [13], such a semiclassical wavefunction continued through a turning point changes its phase by ± 1. (These changes are controlled in general by so-called Maslov indices). Synthetically the whole operation is performed with the help of the Maslov canonical operator [13].

It is easy to note, however, that the necessity to use Fourier transformation disappears if it is possible to somehow avoid turning (caustic) points on the way the wavefunction is continued on. This can be achieved, for example, by enlarging the number of dimensions the problem is formulated in. The complexification of the problem is one of such ways to be used [15]. In the 1D case this can be done effectively and without appealing directly to the semiclassical series expansions by defining the problem on the complex coordinate plane and utilizing the notions of Stokes graphs and fundamental solutions. In comparison with Maslov's approach, the complex coordinate plane (in fact the latter is rather a Riemann surface) corresponds to the complex Lagrangian manifold endowed with the coordinate charts collected of all canonical domains defined by the corresponding Stokes graph. To each canonical domain a fundamental solution is attached having the corresponding domain as the maximal one where its semiclassical expansion as given by (11) is valid. There is no necessity to construct and use the Maslov canonical operator to continue (analytically) the fundamental solutions and to match them in any domain of the plane. The Maslov indices gained by the fundamental solutions on the way of their analytical continuations are provided by crossed cuts of the corresponding Riemann surface. Therefore using the fundamental solution method in the 1D problems is completely equivalent to the corresponding Maslov one in the semiclassical regime of the problem but it has many obvious advantages over the latter with their use as the exact solutions to Schrödinger equation being the first one. Other important properties of the method have been mentioned and used in the main body of this as well as other papers [6–9, 11].

Appendix B

We shall show here that the form (4) of the fundamental solutions as defined by (5)–(7) is independent of the change of variable procedure in the Schrödinger equation. The last method was applied originally by Fröman and Fröman [3].

To this end let us substitute the Dirac form (4) of the solution to the Schrödinger equation (3). Using the notations of section 2 we get the following equation for the χ-factor of (4):

$$(\tilde{q}^{-\frac{1}{2}}(x)\chi'(x))' + \frac{2\sigma}{\hbar}\chi'(x) + \omega(x)\chi(x) = 0 \tag{B.1}$$

where we have dropped for a while the sector index k as irrelevant for the present discussion.

Multiplying now (B.1) by $\exp(\frac{2\sigma}{\hbar}\int_{x_i}^{x}\tilde{q}^{\frac{1}{2}}(\xi)\,\mathrm{d}\xi)$ we get:

$$(\tilde{q}^{-\frac{1}{2}}(x)\chi'(x)\mathrm{e}^{\frac{2\sigma}{\hbar}\int_{x_i}^{x}\tilde{q}^{\frac{1}{2}}(\xi)\,\mathrm{d}\xi})' = -\omega(x)\chi(x)\mathrm{e}^{\frac{2\sigma}{\hbar}\int_{x_i}^{x}\tilde{q}^{\frac{1}{2}}(\xi)\,\mathrm{d}\xi}. \tag{B.2}$$

Taking now some *regular* point y_0 of $\tilde{q}(x, \hbar^2)$ and assuming for $\chi(x)$ the 'initial' condition $\chi(y_0) = \chi_0$ and $\chi'(y_0) = \chi_0'$ we can transform (B.2) into the following integral equation:

$$\chi(x) = \chi_0 + \frac{\sigma\hbar}{2}\tilde{q}_0^{-\frac{1}{2}}\chi_0'(1 - \mathrm{e}^{\frac{-2\sigma}{\hbar}\int_{y_0}^{x}\tilde{q}^{\frac{1}{2}}(\xi)\,\mathrm{d}\xi}) - \frac{\sigma\hbar}{2}\int_{y_0}^{x}(1 - \mathrm{e}^{\frac{-2\sigma}{\hbar}\int_{y_0}^{\xi}\tilde{q}^{\frac{1}{2}}(\eta)\,\mathrm{d}\eta})\omega(\xi)\chi(\xi)\,\mathrm{d}\xi \tag{B.3}$$

with $\tilde{q}(y_0, \hbar^2) = \tilde{q}_0$.

The last equation is general, i.e., the choices of the functions $\delta(x)$ and $f(x)$ as well as of the *regular* point y_0 are arbitrary. The equation can be easily iterated leading us to the functional series for $\chi(x)$ which is convergent uniformly for $\mathcal{R}(\sigma\int_{y_0}^{x}\tilde{q}^{\frac{1}{2}})\,\mathrm{d}\xi > 0$ when the condition (8) for the integration path is satisfied. We can get in this way at each *regular* point y_0 of $\tilde{q}(x, \hbar^2)$ and for a given (but arbitrary) $\sigma = \pm 1$ *two* independent solutions having the Dirac form (4).

To get, however, the fundamental solutions from (B.3) we have to choose instead of the point y_0 a singular point x_0 defining some sector k and and to assume x to lie in this sector as well as we to choose the functions $\delta(x)$ and $f(x)$ according to their constructions in section 2. Then since x_0 is now singular for the actions (7) the exponential function in the second term of the RHS in (B.3) vanishes by this choice whilst the integral remains finite by the constructions of both the Langer term and the function $f(x)$. The consistency condition needs then also to put χ_0' to zero in this case. Putting yet $\chi_0 = 1$ we get the formula (5) by the infinite iteration of (B.3) prepared in this way.

A crucial point for the results of section 3 is to note that at a given sector one can define only *one* fundamental solution at the singular point x_0 of this sector i.e. any two solutions constructed in section 3, vanishing for x approaching x_0 inside the sector and differing by the choice of their f-functions have to coincide with themselves up to some multiplicative constant.

The last conclusion follows easily from the existence, as was mentioned above, of two independent solutions of the Dirac form (4) for any *regular* point y_0. We can take as these solutions the ones constructed for $f(x) \equiv 0$ and satisfying the following two pairs of the 'initial' conditions: $\chi(y_0) = 1$, $\chi'(y_0) = 0$ and $\chi(y_0) = 0$, $\chi'(y_0) = \frac{2\sigma}{\hbar}\bar{q}(y_0) = \frac{2\sigma}{\hbar}\bar{q}_0$ where $\bar{q}(x) = q(x) + \hbar^2\delta(x)$. Then the two independent solutions ϕ_1 and ϕ_2 to the Schrödinger equation (3) corresponding to the last conditions satisfy the following 'initial' conditions:

$$\begin{aligned}
\phi_1(y_0) &= \bar{q}_0^{-\frac{1}{4}}\exp\left(\frac{\sigma}{\hbar}\int_{x_i}^{y_0}\bar{q}^{\frac{1}{2}}(\xi)\,\mathrm{d}\xi\right)\\
\phi_1'(y_0) &= -\frac{1}{4}\bar{q}_0^{-\frac{5}{4}}\bar{q}_0'\exp\left(\frac{\sigma}{\hbar}\int_{x_i}^{y_0}\bar{q}^{\frac{1}{2}}(\xi)\,\mathrm{d}\xi\right) + \frac{\sigma}{\hbar}\bar{q}_0^{\frac{1}{4}}\exp\left(\frac{\sigma}{\hbar}\int_{x_i}^{y_0}\bar{q}^{\frac{1}{2}}(\xi)\,\mathrm{d}\xi\right)\\
\phi_2(y_0) &= 0 \qquad \phi_2'(y_0) = \frac{2\sigma}{\hbar}\bar{q}_0^{\frac{1}{4}}\exp\left(\frac{\sigma}{\hbar}\int_{x_i}^{y_0}\bar{q}^{\frac{1}{2}}(\xi)\,\mathrm{d}\xi\right)
\end{aligned} \tag{B.4}$$

respectively.

Then a fundamental solution $\Psi_k(x)$ corresponding to the choice of some $f(x) \neq 0$ is a linear combination of ϕ_1 and ϕ_2 given by:

$$\Psi_k(x) = C_1\phi_1 + C_2\phi_2 \tag{B.5}$$

where C_1 and C_2 satisfy the equations:

$$\begin{gathered}
C_1(\overline{h}) = \left(1 + \overline{h}^2 \frac{f(y_0)}{\bar{q}(y_0)}\right)^{-\frac{1}{4}} \exp\left[\sigma\overline{h}\int_{x_i}^{y_0} \frac{f(x)}{\sqrt{\bar{q}(x)} + \sqrt{\tilde{q}(x)}}\,\mathrm{d}x\right]\chi_k(y_0) \\
\left[\left(-\frac{1}{4}\tilde{q}_0^{-\frac{5}{4}}\tilde{q}_0' + \frac{\sigma}{\overline{h}}\tilde{q}_0^{\frac{1}{4}}\right)\chi_k(y_0) + \tilde{q}_0^{-\frac{1}{4}}\chi_k'(y_0)\right]\exp\left[\sigma\overline{h}\int_{x_i}^{y_0} \frac{f(x)}{\sqrt{\bar{q}(x)} + \sqrt{\tilde{q}(x)}}\,\mathrm{d}x\right] \\
= \left(-\frac{1}{4}\bar{q}_0^{-\frac{5}{4}}\bar{q}_0' + \frac{\sigma}{\overline{h}}\bar{q}_0^{\frac{1}{4}}\right)C_1 + \frac{2\sigma}{\overline{h}}\bar{q}_0^{\frac{1}{4}}C_2.
\end{gathered} \tag{B.6}$$

It follows now from the last equations that when y_0 approaches the singular point x_0 then C_1 approaches the value given by (15) whilst C_2 vanishes. On the other hand it is easy to check that both the χ-factors coresponding to ϕ_1 and ϕ_2 have the same limit when $y_0 \to x_0$ (they are the solutions to the same integral equation (B.3) which comes out when $y_0 \to x_0$). This limit is given by (5).

Appendix C

We shall show here that the property of the fundamental solutions to be Borel summable in their sectors established earlier for the polynomial potentials [8] is valid for the meromorphic ones as well. For simplicity, we shall demonstrate it for some simple but representative potentials since the basic method of the proof can be also applied to a general meromorphic potential.

The potentials we are going to consider are the following:

$$\begin{aligned}
&V_1(x) = -\frac{\alpha}{x} + \frac{\beta}{x^2} \\
&V_2(x) = -\frac{\alpha}{x} + \frac{\beta}{x^2} + \frac{\gamma}{x^3} \\
&V_3(x) = -\frac{\alpha}{x} + \frac{\beta}{x^4} \\
&\alpha, \quad \beta, \quad \gamma > 0.
\end{aligned} \tag{C.1}$$

Choosing the above signs of $\alpha,\ \beta,\ \gamma$ we want to create a possibility for the bound states to exist, considering this as a rather typical situation for the kind of problems being investigated.

Let us consider first the potential $V_2(x)$. Conclusions which follow from this case shall be shown by considering the fourth-order pole to be independent of the pole order. The only exception is the case of the double pole which has to be considered separately.

The potential $V_2(x)$ is shown in Figure 2(*a*). Assuming the 'ecomomic' $q(x)(= V_2(x) - E)$ corresponding to it with its negative energy E lying, however, above the right local minimum of $V_2(x)$ we get the corresponding Stokes graph on the x-plane shown in figure 2(*b*).

It is now convenient to make the Langer substitution $x = \mathrm{e}^y$ to move the singularity of $V_2(x)$ at $x = 0$ to the left infinity of the y-plane and to consider an equivalent problem on the latter plane with the new $Q(y, \overline{h}^2)$ given, according to (16), by:

$$Q(y) = -\alpha\mathrm{e}^y + \gamma\mathrm{e}^{-y} - E\mathrm{e}^{2y} + \beta + \frac{\overline{h}^2}{4} \tag{C.2}$$

and with the corresponding Stokes graph shown in figure 3.

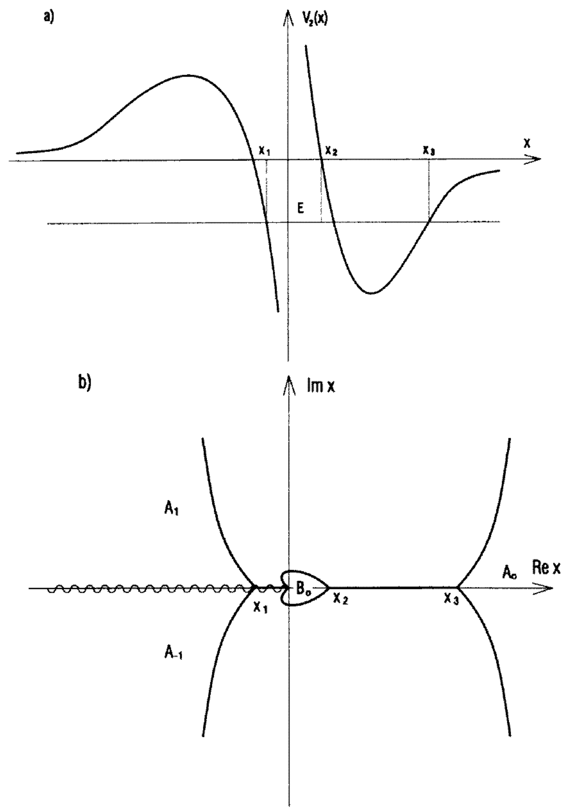

Figure 2. The potential $V_2(x)$ (a), and the corresponding Stokes graph (b).

The fundamental solutions (as well as all other solutions) corresponding to the potential $V_2(x)$ are defined on an infinitely sheeted Riemann surface branching at $x = 0$. The Stokes graph of figure 2(b) corresponds to one of the sheets cut along the real negative half-axis. On the remaining sheets, however, the Stokes graph looks the same.

There are four sectors on each sheet (A_{-1}, A_0, A_1 and B_0 on the sheet of figure 2(b)). The two of them (A_{-1} and A_1) touching the corresponding cut continue (in opposite directions) to the neighbouring two sheets.

Therefore there are infinitely many sectors of the above two types (A and B) on the considered Riemann surface. Under the Langer transformation $y = \log x$ this Riemann surface unfolds into the single y-plane with the images of the sectors of the sheet of figure 2(b) shown in figure 3 (the corresponding sectors being denoted by the same characters). The sectors of the remaining sheets are distributed periodically on the y-plane.

It is obvious that the fundamental solutions constructed on the y-plane according to the Stokes graph of figure 3 are in one-to-one correspondence with the respective fundamental solutions on the x-Riemann surface (the latter can be obtained from the former by formula (17)) and this correspondence coincides with the one between the sectors. Thus in this way the fundamental solutions corresponding to all the sectors of the type B_0 of figure 2(b) correspond to the fundamental solutions defined in all the sectors lying in the left half of the y-plane whilst

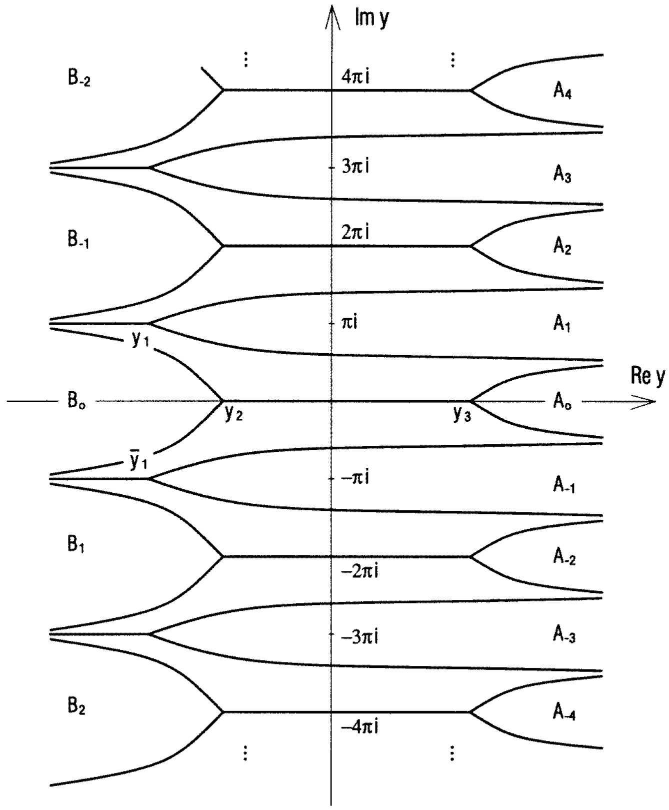

Figure 3. The Stokes graph corresponding to $q_2(x) = V(x) - E$ after the subtitution $x = e^y$.

the remaining fundamental solutions of the x-Riemann surface (of the type A_{-1}, A_0 and A_1) are defined in the sectors of the y-plane lying in the right half of the plane.

It follows further from (17) that the Borel summability of the fundamental solution defined on the x-Riemann surface is equivalent to the corresponding summability of the fundamental solutions defined on the y-plane. The Borel summability of the latter solutions follows, however, directly from the Stokes graph of figure 3.

To show this, let us first note that all the sectors as well as the fundamental solutions defined on the x-Riemann surface were unified by the Langer transformation i.e. on the y-plane their properties are similar. In particular they behave similarly under the rotation in the $\overline{h}$-plane around the point $\overline{h} = 0$. Namely, the asymptotes of the Stokes lines running to the left (right) infinity of the y-plane move down (up) by 2ϕ (ϕ) when $\arg\overline{h}$þ increases by ϕ whilst all the Stokes lines rotate in vicinities of turning poinst around these points by the angle $2\phi/3$. All these motions have opposite directions when $\arg\overline{h}$ decreases by ϕ.

It is now easy to realize that for $\phi = \pi$ (with y kept fixed) the sectors as well as the fundamental solutions corresponding to them transform into themselves according to the rules: $\ldots \to A_{-n} \to A_{-n+1} \to \ldots \to A_{-1} \to A_0 \to A_1 \to \ldots \to A_{n-1} \to A_n \to \ldots$ and $\ldots \to B_{-n} \to B_{-n+1} \to \ldots \to B_{-1} \to B_0 \to B_1 \to \ldots \to B_{n-1} \to B_n \to \ldots$. The arrows in the last transformations are to be reversed if $\phi = -\pi$.

However, this is exactly the situation met in the polynomial potential case [8] and therefore all the conclusions and constructions done for this case in the quoted paper (i.e.

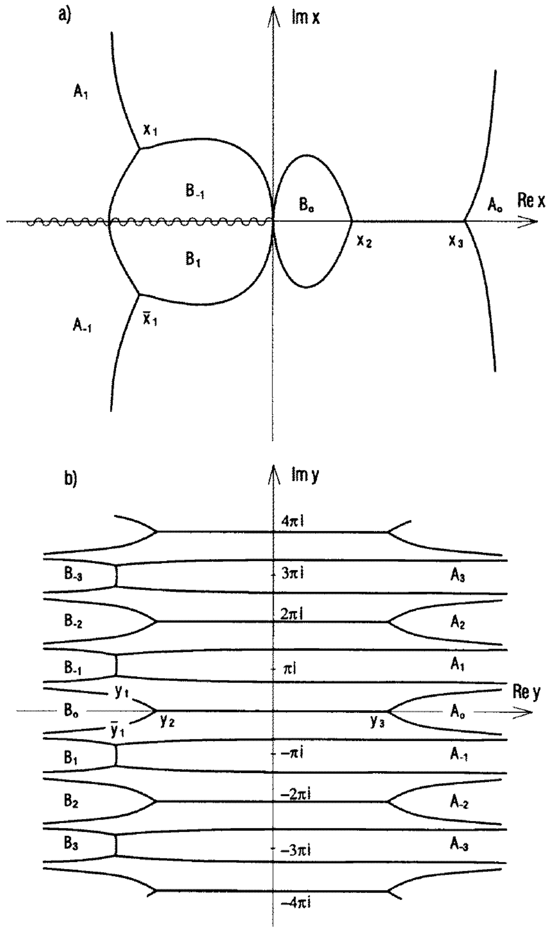

Figure 4. The Stokes graph corresponding to $V_3(x)$ in the x-plane (a), and in the y-plane after transformation $x = \mathrm{e}^y$ (b).

the holomorphicity of each fundamental solution with respect to $\overline{h}$ in a sector $H = \{\overline{h} : |\overline{h}| < \overline{h}_0, |\phi| < \pi\}$, the calculation of its jump on the cut $C = \{\overline{h} : |\overline{h}| < \overline{h}_0, |\phi| = \pi\}$ and the applying of the Bender–Wu formula to calculate the (factorial) rate of grow of coefficients of its semiclassical expansion) are valid here as well.

Therefore, our main conclusion for the case considered is that all their fundamental solutions when expanded semiclassically in the sectors defining them are recovered by the Borel resummations.

Consider now the potential $V_3(x)$ and $q(x) = V_3(x) - E$, with $0 > E > V_{3,min}$ and make the Langer change of variable $x = \mathrm{e}^y$ to get the resulting $Q(y)$ as $Q(y) = -\alpha\mathrm{e}^{-y} + \beta\mathrm{e}^{-y} - E\mathrm{e}^{2y} + \frac{\overline{h}^2}{4}$. The Stokes graphs corresponding to both $q(x)$ and $Q(y)$ are then shown in figure 4. It follows from the figure that the arguments applied previously to the second of the potentials (C.1) are obviously still valid in the considered case and we can again claim that the semiclassical expansions of the fundamental solutions corresponding to the sectors A_n and B_n, $-\infty < n < +\infty$, on both the figures 4a and 4b (the sectors on the figures denoted identically map into each other by the Langer transform $x = \mathrm{e}^{y}$) are Borel summable to the solutions themselves.

The case of the double pole potential is exceptional since as it is seen on figures 5(*a*) and (*b*) showing the Stokes graphs corresponding to $q(x) = -\alpha x^{-1} + \beta x^{-2} - E$ and $Q(y) = -\alpha \mathrm{e}^{-y} - E\mathrm{e}^{2y} + \beta + \frac{\hbar^2}{4}$ $(0 > E > V_{1,\min})$, respectively, there is only *one* fundamental solution of the B-type (defined at the double pole) and infinitely many fundamental solutions of the A-type. The semiclassical expansions of the latter are obviously Borel summable (to the solutions themselves) whilst the Borel summability of the unique B-type solution is unclear at first sight.

Namely, looking at figure 5(*c*) which represents the Stokes graph of figure 5(*b*) rotated by $\arg\overline{h} = \pm\pi/2$ we see that there are two fundamental solutions which can be defined in the half plane lying to the left from the vertical (anti-Stokes) line joining the turning point y_1 with all its periodic displacements. One of these solutions is constructed along the vertical canonical path $\gamma_u(y)$ emerging from the 'upper' infinity of the y-plane (see figure 5(*c*)) and running downwards to the point y on the figure, whilst the second solution is defined on the analogous path $\gamma_d(y)$ running upwards. These solutions, by their construction, have well-defined semiclassical expansions (11) and are limiting solutions of the one defined by the canonical path $\gamma(y)$ in the sector B of Stokes graph of figure 5(*b*) when the latter graph is rotated by $\arg\overline{h} = \pm\pi/2$.

Therefore we can conclude that the Nevanlinna–Watson–Sokal theorem for the unique fundamental solution of sector B are satisfied with respect to the desired domain of holomorphicity (coinciding with the sector $H_1 = \{\overline{h} : |\overline{h}| < \overline{h}_0, -\frac{\pi}{2} \leqslant \arg\overline{h} \leqslant \frac{\pi}{2}\}$) and to the desired semiclassical expansion whilst the desired factorial rate of growth of the coefficients of its semiclassical expansion is still not established. Namely, because of the infinite series of the turning points displaced vertically, we cannot rotate the fundamental solution of sector B beyond sector H_1 without destroying simultaneously the analyticity of its representation (4).

Therefore the method of Bender and Wu cannot be applied in this case.

However, we can use the results obtained for the case of the potential $V_2(x)$ to argue that the discussed semiclassical coefficients have the desired factorial growth. Namely, we can note that the fundamental solution Ψ_0 attached to the sector B_0 of figure 3, as well as the coefficients of its semiclassical expansion, approach the solution Ψ of sector B of figure 5(*b*) and its semiclassical coefficients, respectively, when the positive coefficient γ in the potential $V_2(x)$ vanishes. These limits are not uniform in γ, however. Nevertheless, since the semiclassical coefficients of Ψ_0 are known to grow factorially, the same property has to have its limit for $\gamma \to 0_+$. In this way the third demand of the Nevanlinna–Watson–Sokal is satisfied also in this case, proving the Borel summability of *all* the fundamental solutions of the considered case.

An extension of the arguments used in the above discussion to a general meromorphic potential seems to be straightforward (by applying to a chosen particular pole the Langer transform which maps the pole to the (left) infinity of the y-plane) although the unavoidable proliferation of poles and turning points in the y-plane can make the corresponding analysis laborious.

Appendix D

We consider here a simple example of the meromorphic potential $V(x) = \sin^{-2} x$ to illustrate the action of the change-of-variable procedure in its form described at the end of section 5. A z-function generating a corresponding function $y(x)$ can be easily guessed to be $z(x) = \sin x$

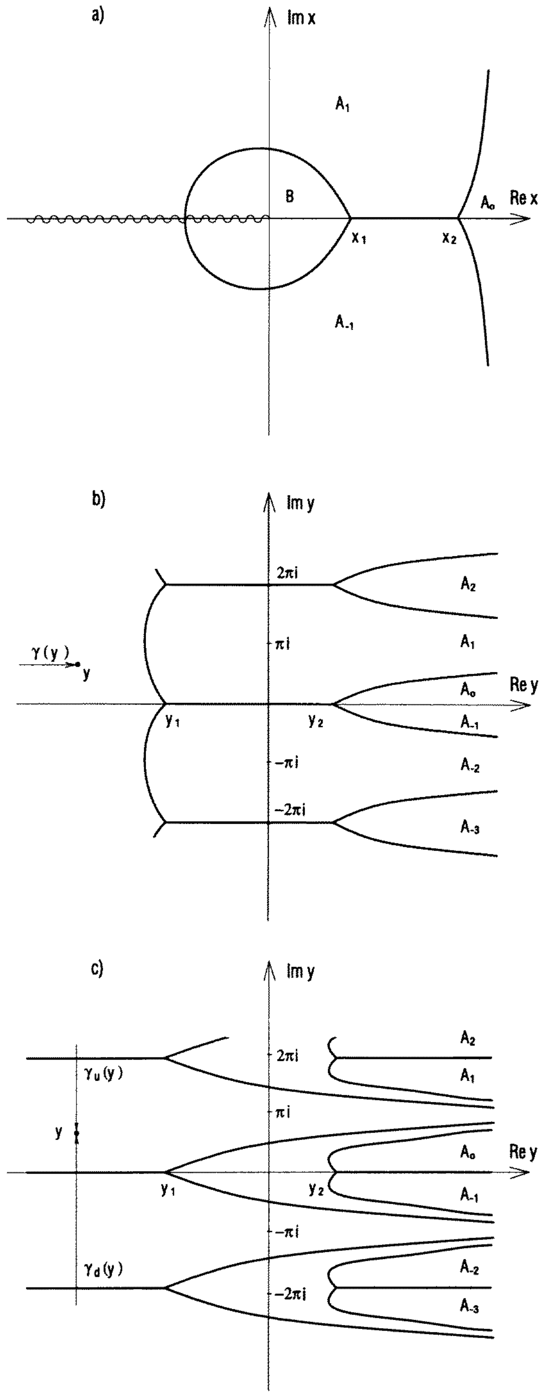

Figure 5. The Stokes graph corresponding to $V_1(x)$: (a) in the x-plane, (b) in the y-plane and (c) in the y-plane rotated by $\arg \hbar = \pm\pi/2$.

and therefore $y(x) = \log \sin x$. Then a modification of $q(x) = V(x) - E$ to $\tilde{q}(x, \overline{h}^2)$ by the Schwarzian of $y(x)$ gives:

$$\tilde{q}(x, \overline{h}^2) = \frac{1}{\sin^2 x} + \frac{\overline{h}^2}{4}\frac{1}{\sin^2 x} - \frac{3\overline{h}^2}{4}\frac{1}{\cos^2 x}. \tag{D.1}$$

One can easily recognize in (D.1) the assumed functional and meromorphic structure of (1). The forms of $x'^2(y) = \mathrm{e}^{2y}/(1 - \mathrm{e}^{2y})$ and of $q(x(y)) = \mathrm{e}^{-2y} - E$ also satisfy the corresponding assumptions about the transformation $x(y)$ done in section 5.

We can of course now discard the f-term in (D.1) (equal to $-3/(4\cos^2 x)$) to obtain the resulting $\tilde{q}(x, \overline{h}^2)$ in the form of Bailey [20].

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